

# Suslin's moving lemma with modulus

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## Abstract

The moving lemma of Suslin states that a cycle on  $X \times \mathbb{A}^n$  meeting all faces properly can be moved so that it becomes equidimensional over  $\mathbb{A}^n$ . This leads to an isomorphism of motivic Borel-Moore homology and higher Chow groups.

In this short paper we formulate and prove a variant of this. It leads to an isomorphism of Suslin homology with modulus and higher Chow groups with modulus, in an appropriate pro setting.

## 1 Introduction

Suslin [Sus] has proved roughly that a cycle on  $X \times \mathbb{A}^n$  meeting all faces properly can be moved so that it becomes equidimensional over  $\mathbb{A}^n$ . Here  $X$  is an affine variety over a base field  $k$ . As a consequence he obtains that the inclusion ( $r \geq 0$ )

$$z_r^{equi}(X, \bullet) \hookrightarrow z_r(X, \bullet)$$

of the cycle complex of equidimensional cycles into Bloch's cycle complex is a quasi-isomorphism.

Recently the context has been extended to cycles with modulus by Binda-Kerz-Saito [KS, BS] and Kahn-Saito-Yamazaki [KSY]. The reader finds the definitions below. There is an obvious injection ( $r \geq 0$ )

$$z_r^{equi}(\overline{X}|Y, \bullet) \hookrightarrow z_r(\overline{X}|Y, \bullet)$$

for each pair  $(\overline{X}, Y)$  consisting of a finite type  $k$ -scheme  $\overline{X}$  and an effective Cartier divisor  $Y$  on it. We usually write  $X := \overline{X} \setminus Y$ .

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In this paper we prove a variant of Suslin's moving lemma which takes the modulus condition into account (Theorem 2.8 below). Our version of Suslin's moving lemma implies the following:

**Theorem 1.1** (Theorem 3.1). *Suppose  $\overline{X}$  is affine and  $X$  is an open set of  $\overline{X}$  such that  $\overline{X} \setminus X$  is the support of an effective Cartier divisor  $Y$ . Let  $r \geq 0$ . Then the inclusions ( $m \geq 0$ )*

$$z_r^{equi}(\overline{X}|mY, \bullet) \subset z_r(\overline{X}|mY, \bullet)$$

*induce an isomorphism of abelian groups*

$$\varprojlim_m H_n(z_r^{equi}(\overline{X}|mY, \bullet)) \cong \varprojlim_m \mathrm{CH}_r(\overline{X}|mY, n).$$

Actually we can prove an isomorphism of pro-abelian groups. We do not know if the inclusion is a quasi-isomorphism before taking limits.

Now we recall the definitions. We set  $\square^n := (\mathbb{P}^1 \setminus \{\infty\})^n = \mathrm{Spec}(k[y_1, \dots, y_n])$  in this paper, contradicting some authors who prefer 1 as the point at infinity. With this convention our computations look simpler. We set a divisor on  $(\mathbb{P}^1)^n$ :

$$F_n = \sum_{i=1}^n (\mathbb{P}^1)^{i-1} \times \{\infty\} \times (\mathbb{P}^1)^{n-i}.$$

The *faces* of  $\square^n$  are  $\{y_i = 0\}$ ,  $\{y_i = 1\}$  and their intersections.

**Definition 1.2** ([BS], [KSY]). (1) Let  $\underline{z}_r(\overline{X}|Y, n)$  be the group of  $(r+n)$ -dimensional cycles on  $X \times \square^n$  whose components  $V$  meet faces of  $\square^n$  properly, and *have modulus  $Y$* , i.e.:

Let  $\overline{V}^N$  be the normalization of  $\overline{V} \subset \overline{X} \times (\mathbb{P}^1)^n$ , the closure of  $V$ . Let  $\varphi_V: \overline{V}^N \rightarrow \overline{X} \times (\mathbb{P}^1)^n$  be the natural map. Then the inequality of Cartier divisors

$$\varphi_V^{-1}(Y \times (\mathbb{P}^1)^n) \leq \varphi_V^{-1}(\overline{X} \times F_n)$$

holds. (When  $n = 0$  the condition reads: the closure  $\overline{V} \subset \overline{X}$  of  $V$  is contained in  $X$  i.e.  $V = \overline{V}$ .)

Let  $\partial_{i,0}: \square^{n-1} \hookrightarrow \square^n$  be the embedding of the face  $\{y_i = 0\}$ :

$$\partial_{i,0}: (y_1, \dots, y_{n-1}) \mapsto (y_1, \dots, \overset{i}{0}, y_i, \dots, y_{n-1}).$$

Define  $\partial_{i,1}$  similarly. The groups  $\underline{z}_r(\overline{X}|Y, n)$  form a complex by the differentials

$$\sum_{i=1}^n (-1)^i (\partial_{i,1}^* - \partial_{i,0}^*): \underline{z}_r(\overline{X}|Y, n) \rightarrow \underline{z}_r(\overline{X}|Y, n-1).$$

(2) Let  $\underline{z}_r^{equi}(\overline{X}|Y, n)$  be the subgroup of  $\underline{z}_r(\overline{X}|Y, n)$  consisting of cycles that are equidimensional over  $\square^n$  (necessarily of relative dimension  $r$ ).

**Remark 1.3.** The condition that  $V$  has modulus  $Y$  makes sense for any closed subset  $V$  of  $X \times \square^n$ . (In that case, normalization of a closed subset means the disjoint union of the normalizations of its reduced irreducible components.)

**Definition 1.4.** We define the *degenerate part*  $\underline{z}_r(\overline{X}|Y, n)_{\text{degn}} \subset \underline{z}_r(\overline{X}|Y, n)$  as the subgroup generated by the cycles of the form  $(\text{id}_{\overline{X}} \times \text{pr}_i)^*(V)$ , where  $V \in \underline{z}_r(\overline{X}|Y, n-1)$  and  $\text{pr}_i : \square^n \rightarrow \square^{n-1}$ ,  $(y_1, \dots, y_n) \mapsto (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$  for some  $i = 1, \dots, n$ . We also define the *degenerate part*  $\underline{z}_r^{\text{equi}}(\overline{X}|Y, n)_{\text{degn}} \subset \underline{z}_r^{\text{equi}}(\overline{X}|Y, n)$  in a similar way. We set

$$\begin{aligned} z_r(\overline{X}|Y, n) &:= \underline{z}_r(\overline{X}|Y, n) / \underline{z}_r(\overline{X}|Y, n)_{\text{degn}}, \\ z_r^{\text{equi}}(\overline{X}|Y, n) &:= \underline{z}_r^{\text{equi}}(\overline{X}|Y, n) / \underline{z}_r^{\text{equi}}(\overline{X}|Y, n)_{\text{degn}}. \end{aligned}$$

Noting that the differentials  $\partial_{i,0}, \partial_{i,1}$  preserve degenerate parts, we can see that  $z_r(\overline{X}|Y, n)$  and  $z_r^{\text{equi}}(\overline{X}|Y, n)$  also form complexes. We define the *higher Chow group with modulus* by

$$\text{CH}_r(\overline{X}|Y, n) := H_n(z_r(\overline{X}|Y, \bullet)).$$

**Remark 1.5.** The subgroups

$$\underline{z}_r(\overline{X}|Y, n)_0 := \bigcap_{i=1}^n \ker(\partial_{i,0}^*) \subset \underline{z}_r(\overline{X}|Y, n)$$

form a subcomplex. One checks that the composite map

$$\underline{z}_r(\overline{X}|Y, \bullet)_0 \hookrightarrow \underline{z}_r(\overline{X}|Y, \bullet) \rightarrow z_r(\overline{X}|Y, \bullet)$$

is an isomorphism of complexes. Using this, we have a direct sum decomposition

$$\underline{z}_r(\overline{X}|Y, \bullet) = z_r(\overline{X}|Y, \bullet) \oplus \underline{z}_r(\overline{X}|Y, \bullet)_{\text{degn}}$$

of a complex. We have a similar decomposition of  $\underline{z}_r^{\text{equi}}(\overline{X}|Y, \bullet)$ , and the inclusion  $\underline{z}_r^{\text{equi}}(\overline{X}|Y, \bullet) \hookrightarrow \underline{z}_r(\overline{X}|Y, \bullet)$  is compatible with the decompositions.

## 2 Equidimensionality theorem

Let  $k$  be an infinite base field. We will formulate and prove a variant of Suslin's equidimensionality theorem for modulus pairs  $(\overline{X}, Y)$  (i.e. a  $k$ -scheme  $\overline{X}$  equipped with an effective Cartier divisor  $Y$ ) for which  $\overline{X}$  is affine.

### 2.1 Suslin's generic equidimensionality theorem (review)

**Theorem 2.1** ([Sus, Th.1.1]). *Assume that  $\overline{X}$  is an affine scheme,  $V$  is a closed subscheme in  $\overline{X} \times \square^n$  and  $t$  is a nonnegative integer such that  $\dim V \leq n + t$ . Assume further that  $Z$  is an effective divisor in  $\square^n$  and  $\varphi : \overline{X} \times Z \rightarrow \overline{X} \times \square^n$  is any  $\overline{X}$ -morphism. Then there exists an  $\overline{X}$ -morphism  $\Phi : \overline{X} \times \square^n \rightarrow \overline{X} \times \square^n$  such that*

(1)  $\Phi|_{\overline{X} \times Z} = \varphi$

(2) *Fibers of the projection  $\Phi^{-1}(V) \rightarrow \square^n$  over points of  $\square^n \setminus Z$  have dimension  $\leq t$ .*

*Sketch of proof.* Note that the  $\overline{X}$ -morphisms  $\varphi, \Phi$  are determined by  $n$  regular functions on  $\overline{X} \times Z$  and  $\overline{X} \times \square^n$  respectively.

We can reduce the problem to the case  $\overline{X} = \mathbb{A}^m$  as follows. Take any closed embedding  $\overline{X} \hookrightarrow \mathbb{A}^m$  and regard  $V$  as a subset of  $\mathbb{A}^m \times \square^n$ . By the above observation, the given  $\varphi$  can be extended to an  $\mathbb{A}^m$ -morphism  $\varphi: \mathbb{A}^m \times Z \rightarrow \mathbb{A}^m \times \square^n$ . Suppose we have found an  $\mathbb{A}^m$ -morphism  $\Phi: \mathbb{A}^m \times \square^n \rightarrow \mathbb{A}^m \times \square^n$  with the desired properties for  $\mathbb{A}^m$  and  $V$ . It restricts to an  $\overline{X}$ -morphism  $\overline{X} \times \square^n \rightarrow \overline{X} \times \square^n$  and satisfies the desired properties for  $\overline{X}$  and  $V$ .

From now on we assume  $\overline{X} = \mathbb{A}^m$ . Let  $x_1, \dots, x_m$  be the coordinates of  $\mathbb{A}^m$  and  $y_1, \dots, y_n$  be the coordinates of  $\square^n$ . Let  $h(\underline{y})$  be the defining equation of  $Z \subset \square^n$ .

We are given a  $\overline{X}$ -morphism  $\varphi: \overline{X} \times Z \rightarrow \overline{X} \times \square^n$  i.e. a  $k$ -morphism  $\overline{X} \times Z \rightarrow \square^n$ . It corresponds to  $k$ -algebra homomorphisms

$$\begin{aligned} k[y_1, \dots, y_n] &\rightarrow k[x_1, \dots, x_m, y_1, \dots, y_n]/(h(\underline{y})) \\ y_i &\mapsto f_i(\underline{x}, \underline{y}) \mod (h(\underline{y})). \end{aligned}$$

Suslin constructs the desired morphism  $\Phi$  as the morphism corresponding to homomorphisms of the form

$$\begin{aligned} k[y_1, \dots, y_n] &\rightarrow k[x_1, \dots, x_m, y_1, \dots, y_n] \\ y_i &\mapsto \Phi_i := f_i(\underline{x}, \underline{y}) + h(\underline{y})F_i(\underline{x}) \end{aligned}$$

where  $F_1(\underline{x}), \dots, F_n(\underline{x})$  are appropriately chosen homogeneous polynomials of degree  $N$  for a large  $N$ .

He has shown that if we take  $N$  large enough, then a generic choice of  $F_1, \dots, F_n$  makes the equidimensionality condition true.  $\square$

## 2.2 Suslin's equidimensionality theorem, with modulus

Recall a *face* of  $\square^n = \text{Spec}(k[y_1, \dots, y_n])$  is a closed subscheme of the form  $\{y_i = 0\}, \{y_i = 1\}$  or an intersection of them. Put  $\partial\square^n = \cup_{\partial}\partial(\square^{n-1})$  where  $\partial: \square^{n-1} \hookrightarrow \square^n$  runs through embeddings of codimension 1 faces. It is a closed subset defined by the equation  $h(\underline{y}) = y_1(1 - y_1) \dots y_n(1 - y_n)$ .

We need the following version of Suslin's moving lemma where we control the degrees of the map  $\Phi$ .

**Theorem 2.2.** *Let  $\overline{X} = \text{Spec}(R)$  be an affine  $k$ -scheme and  $V \subset \overline{X} \times \square^n$  be a closed subset of dimension  $n + t$  for some  $t \geq 0$ . Suppose given a morphism*

$$\Phi': \overline{X} \times \partial\square^n \rightarrow \overline{X} \times \square^n$$

*and suppose there is an integer  $d \geq 2$  such that for any codimension 1 face  $\partial: \square^{n-1} \hookrightarrow \square^n$ , the composite  $\Phi' \circ (\text{id}_{\overline{X}} \times \partial)$  is defined by polynomials  $\Phi'_{i,\partial} \in$*

$R[y_1, \dots, y_{n-1}]$  ( $1 \leq i \leq n$ ) whose degrees with respect to  $y_j$  are at most  $d$  for each  $j$ .

Then we can find an  $\overline{X}$ -map

$$\Phi^n : \overline{X} \times \square^n \rightarrow \overline{X} \times \square^n$$

extending  $\Phi'$  such that  $(\Phi^n)^{-1}(V) \subset \overline{X} \times \square^n$  has fibers of dimension  $\leq t$  over  $\square^n \setminus \partial \square^n$ , and moreover, the functions  $\Phi_i^n \in R[y_1, \dots, y_n]$  defining  $\Phi^n$  ( $1 \leq i \leq n$ ) have degrees  $\leq d$  with respect to each  $y_j$ .

*Proof.* The map  $\Phi'$  is determined by  $R$ -coefficient polynomials  $f_i(y_1, \dots, y_n) \bmod h(\underline{y})$  ( $1 \leq i \leq n$ ). If we substitute  $y_j = 0$  or  $y_j = 1$  to  $f_i$  we get a polynomial which has degree  $\leq d$  with respect to each  $y_k$  by the hypothesis.

**Lemma 2.3.** *Let  $d \geq 1$  be an integer. Suppose given a polynomial  $f(y_1, \dots, y_n) \in R[y_1, \dots, y_n]$  such that for each  $j$  if we substitute any of  $y_j = 0$  or  $y_j = 1$ , the resulting polynomial has degree  $\leq d$  with respect to each  $y_k$ . Then  $f \bmod h(\underline{y})$  has a (unique) representative which has degree  $\leq d$  with respect to each  $y_j$  (where we keep the notation  $h(\underline{y}) := y_1(1 - y_1) \cdots y_n(1 - y_n)$ ).*

*Proof.* For each  $i$  denote by  $y_i(-|_{y_i=1})$  the operator which sends a polynomial  $f$  to  $y_i \cdot (f|_{y_i=1})$  and define  $(1 - y_i)(-|_{y_i=0})$  similarly. Note that for different  $i$  and  $j$  the operators  $y_i(-|_{y_i=1})$  and  $y_j(-|_{y_j=1})$  commute (and similarly for other pairs). Put  $\alpha_i := 1 - y_i(-|_{y_i=1}) - (1 - y_i)(-|_{y_i=0})$ . Then one can see the polynomial

$$f - (\alpha_1 \dots \alpha_n f)$$

is the desired representative.  $\square$

By the previous lemma, we can replace representatives  $f_i(\underline{y})$  so that they have degrees  $\leq d$  with respect to each  $y_j$ .

By Suslin's proof of Theorem 2.1, there are elements  $F_i \in R$  such that if we define  $\Phi^n$  by setting its components as ( $1 \leq i \leq n$ )

$$\Phi_i^n(\underline{y}) := f_i(\underline{y}) + h(\underline{y})F_i,$$

then the condition on fiber dimensions is satisfied. Moreover, from this form,  $\Phi_i^n$  has degree  $\leq d$  with respect to each  $y_j$ . This completes the proof of Theorem 2.2.  $\square$

**Lemma** (Containment Lemma, [KP, Prop.2.4]). *Let  $V \subset \overline{X} \times \square^n$  be a closed subset which has modulus  $Y$  and  $V' \subset V$  be a smaller closed subset. Then  $V'$  also has modulus  $Y$ .*

**Proposition 2.4.** *Let  $(\overline{X}, Y)$  be a modulus pair with  $\overline{X} = \text{Spec}(R)$  affine. Let  $d$  be a positive integer and  $V \subset \overline{X} \times \square^n$  be a closed subset having modulus  $nd \cdot Y$ . Suppose  $\Phi: \overline{X} \times \square^{n'} \rightarrow \overline{X} \times \square^n$  is an  $\overline{X}$ -morphism defined by polynomials  $\Phi_j \in R[y_1, \dots, y_{n'}]$  ( $1 \leq j \leq n$ ) having degrees  $\leq d$  with respect to each  $y_i$ . Then the closed subset  $\Phi^{-1}(V)$  of  $\overline{X} \times \square^{n'}$  has modulus  $Y$ .*

*Proof.* Since the assertion is local, we may assume  $Y$  is principal and defined by  $u \in R$ . Let  $V'$  denote any one of the irreducible components of  $\Phi^{-1}(V)$  and let  $\overline{V'}^N$  be its normalization of its closure in  $\overline{X} \times (\mathbb{P}^1)^{n'}$ .

$$\begin{array}{ccccccc} \overline{V'}^N & & & & & & \\ \downarrow & & & & & & \\ \overline{V'} & \supset & V' & \subset & \Phi^{-1}(V) & \subset & \overline{X} \times \square^{n'} \\ & & & & \downarrow & & \downarrow \Phi \\ & & & & V & \subset & \overline{X} \times \square^n \end{array}$$

Thanks to Containment Lemma above, the closure of  $\Phi(V') \subset V$  has modulus  $ndY$ . By replacing  $V$  by the closure of  $\Phi(V')$ , we may assume the map  $V' \rightarrow V$  is dominant.

**Claim 2.5.** Let  $\overline{V'}^{N^\circ}$  be the domain of definition of the rational map

$$\overline{V'}^N \rightarrow \overline{X} \times (\mathbb{P}^1)^{n'} \xrightarrow{\Phi} \overline{X} \times (\mathbb{P}^1)^n.$$

Then the complement of  $\overline{V'}^{N^\circ}$  in  $\overline{V'}^N$  has codimension  $\geq 2$ .

*Proof.* Let  $v$  be a point of  $\overline{V'}^N$  of codimension 1. Since the generic point  $\eta$  of  $\overline{V'}^N$  lands on  $\overline{X} \times \square^n$  we have a commutative diagram

$$\begin{array}{ccc} \eta & \in \text{Spec } \mathcal{O}_v & \supset \overline{X} \times (\mathbb{P}^1)^{n'} \\ & & \searrow \Phi \\ & & \supset \overline{X} \times (\mathbb{P}^1)^n \supset \overline{X} \end{array}$$

The assertion follows from the valuative criterion of properness (of the projection  $\overline{X} \times (\mathbb{P}^1)^n \rightarrow \overline{X}$ ).  $\square$

By Claim 2.5, we find that a Cartier divisor on  $\overline{V'}^N$  is effective if and only if its restriction to  $\overline{V'}^{N^\circ}$  is effective, since  $\overline{V'}^N$  is normal.

Write  $\text{pr}_j : \overline{X} \times (\mathbb{P}^1)^n \rightarrow \mathbb{P}^1$  for the projection to the  $j$ -th  $\mathbb{P}^1$  and  $\Phi_j$  for the composite rational map  $\overline{X} \times (\mathbb{P}^1)^{n'} \xrightarrow{\Phi} \overline{X} \times (\mathbb{P}^1)^n \xrightarrow{\text{pr}_j} \mathbb{P}^1$ , also seen as a rational function on  $\overline{X} \times (\mathbb{P}^1)^n$ . We will denote the pull-backs of  $\Phi$  and  $\Phi_j$  to  $\overline{V'}^{N^\circ}$  by  $\Phi^V$  and  $\Phi_j^V$ . By definition of  $\overline{V'}^{N^\circ}$  they are well-defined morphisms from  $\overline{V'}^{N^\circ}$  to  $\overline{X} \times (\mathbb{P}^1)^n$  and to  $\mathbb{P}^1$  respectively. There is a uniquely induced morphism  $\overline{V'}^{N^\circ} \rightarrow \overline{V'}^N$  because now we are assuming  $V' \rightarrow V$  is dominant.

For any given point of  $\overline{V'}^{N^\circ}$ , we can find an affine open set  $\text{Spec}(A) \subset \overline{V'}^N$  and an affine neighborhood  $\text{Spec}(B) \subset \overline{V'}^{N^\circ}$  of the point such that  $\Phi^V$  restricts

to a morphism  $\Phi^V : \text{Spec}(B) \rightarrow \text{Spec}(A)$ .

$$\begin{array}{ccc} \text{Spec}(B) & \subset \overline{V'}^{N^\circ} & \rightarrow \overline{X} \times (\mathbb{P}^1)^{n'} \\ & \downarrow \Phi^V & \downarrow \Phi \\ \text{Spec}(A) & \subset \overline{V}^N & \rightarrow \overline{X} \times (\mathbb{P}^1)^n \end{array}$$

By shrinking  $\text{Spec}(A)$  if necessary, we may assume  $y_j$  or  $1/y_j$  is regular on  $\text{Spec}(A)$  for each  $j$ . Denote by  $J \subset \{1, \dots, n\}$  the set of  $j$ 's for which  $1/y_j$  is regular. The divisor  $F_n$  is defined by the equation  $\frac{1}{\prod_{j \in J} y_j} = 0$  on  $\text{Spec}(A)$ . Since  $V$  has modulus  $ndY$ , the rational function  $\frac{1}{\prod_{j \in J} y_j} / u^{nd}$  on  $\text{Spec}(A)$  is regular. Pulling it back by  $\Phi^V$ , we find that the rational function  $\frac{1}{\prod_{j \in J} \Phi_j^V} / u^{nd}$  on  $\text{Spec}(B)$  is regular.

Shrinking  $\text{Spec}(B)$  if necessary, we may assume  $y_i$  or  $1/y_i$  is regular on  $\text{Spec}(B)$  for each  $i$ . Let  $I \subset \{1, \dots, n'\}$  be the set of  $i$ 's for which  $1/y_i$  is regular on  $\text{Spec}(B)$ ; the divisor  $F_{n'}$  is defined by  $\frac{1}{\prod_{i \in I} y_i} = 0$  on  $\text{Spec}(B)$ .

**Claim 2.6.** The rational function  $\frac{\Phi_j^V}{\prod_{i \in I} y_i^d}$  on  $\text{Spec}(B)$  is regular for each  $j \in \{1, \dots, n\}$  (i.e. it is a morphism from  $\text{Spec}(B)$  into  $\mathbb{A}^1 \subset \mathbb{P}^1$ ).

*Proof.* The function is the restriction of the meromorphic function  $\frac{\Phi_j}{\prod_{i \in I} y_i^d}$  on  $\overline{X} \times (\mathbb{P}^1)^{n'}$ . It is written as an  $R$ -coefficient polynomial in the variables  $1/y_i$  ( $i \in I$ ) and  $y_i$  ( $i \in I^c$ ) by the assumption on  $\Phi$ . So it is regular around the (image of the) considered point on  $\overline{X} \times (\mathbb{P}^1)^{n'}$ .  $\square$

By Claim 2.6 the function

$$\left( \frac{1}{\prod_{j \in J} \Phi_j^V} / u^{nd} \right) \cdot \prod_{j \in J} \frac{\Phi_j^V}{\prod_{i \in I} y_i^d} = \frac{1}{\prod_{i \in I} y_i^{d \cdot \#J}} / u^{nd}$$

is regular on  $\text{Spec}(B)$ . This shows a relation of Cartier divisors on  $\text{Spec}(B)$ :

$$nd \left( \prod_{i \in I} \frac{1}{y_i} \right) - nd(u) \geq 0$$

which implies the relation

$$(\text{pullback of } F_{n'}) - (\text{pullback of } Y) \geq 0$$

on  $\text{Spec}(B)$ , hence on  $\overline{V'}^{N^\circ}$ , which is valid on  $\overline{V'}^N$  as well by the comment made after Claim 2.5. This completes the proof of Proposition 2.4.  $\square$

**Remark 2.7.** Under the hypotheses of Proposition 2.4, we can prove that the morphism  $\Phi$  is admissible [KSY, Def.1.1] for the pairs  $((\mathbb{P}_R^1)^{n'}, ndF_{n'})$ ,  $((\mathbb{P}_R^1)^n, F_n)$ .

It gives an alternative proof of Proposition 2.4. We sketch the proof of the admissibility. We use the fact that admissibility can be checked after replacing the source by an open cover, or after blowing up  $(\mathbb{P}^1)^{n'}$  by a closed subset outside  $\square^{n'}$ . Set  $\eta_i = 1/y_i$ . The scheme  $(\mathbb{P}^1)^{n'}$  is covered by open subsets  $U_I = \text{Spec}(R[\eta_i, y_{i'} \mid i \in I, i' \notin I])$  where  $I$  runs through the subsets of  $\{1, \dots, n'\}$ . On the region  $U_I$ , the rational function  $\Phi_j$  is written as the ratio of the next two regular functions, by the assumption on  $\Phi_j$ .

$$\Phi_j = \frac{\Phi_j^{(I)}(\eta_i, y_{i'})}{\prod_{i \in I} \eta_i^d}.$$

We blow up  $U_I$  by the ideal  $(\Phi_j^{(I)}, \prod_{i \in I} \eta_i^d)$ . We perform this blow up for all  $j \in \{1, \dots, n\}$ . The resulting scheme is covered by  $2^n$  open subsets

$$U_{IJ} = \text{Spec}\left(R\left[\eta_i, y_{i'} \mid i \in I, i' \notin I, \frac{\prod_{i \in I} \eta_i^d}{\Phi_j^{(I)}(\eta_i, y_{i'})}, \frac{\Phi_{j'}^{(I)}(\eta_i, y_{i'})}{\prod_{i \in I} \eta_i^d} \mid j \in J, j' \notin J\right]\right)$$

where  $J$  runs through the subsets of  $\{1, \dots, n\}$ . The morphism  $\Phi$  naturally extends to a morphism  $\Phi: U_{IJ} \rightarrow U_J \subset (\mathbb{P}^1)^n$ .

On  $U_{IJ}$ , the pull-back of  $F_n$  by  $\Phi$  is represented by the function  $\prod_{j \in J} \frac{\prod_{i \in I} \eta_i^d}{\Phi_j^{(I)}(\eta_i, y_{i'})}$ . The divisor  $ndF_{n'}$  is represented by  $\prod_{i \in I} \eta_i^{nd}$ . Their ratio is

$$\prod_i \eta_i^{(n-\#J)d} \cdot \prod_{j \in J} \Phi_j^{(I)}$$

which is a regular function on  $U_{IJ}$ . This proves the admissibility.

From Theorem 2.2 and Proposition 2.4, we get:

**Theorem 2.8.** *Let  $(\overline{X}, Y)$  be a modulus pair with  $\overline{X}$  affine, and  $V \subset \overline{X} \times \square^n$  be a purely  $(n+t)$ -dimensional closed subset for some  $t \geq 0$ . Suppose  $V$  has modulus  $2n \cdot Y$ . Then there is a series of maps*

$$\Phi^\bullet: \overline{X} \times \square^\bullet \rightarrow \overline{X} \times \square^\bullet$$

*compatible with face maps i.e. for any codimension 1 face  $\partial: \square^m \hookrightarrow \square^{m+1}$ , the following commutes:*

$$\begin{array}{ccc} \overline{X} \times \square^m & \xrightarrow{\Phi^m} & \overline{X} \times \square^m \\ \downarrow \partial & & \downarrow \partial \\ \overline{X} \times \square^{m+1} & \xrightarrow{\Phi^{m+1}} & \overline{X} \times \square^{m+1} \end{array}$$

*such that the closed subset*

$$(\Phi^n)^{-1}(V) \subset \overline{X} \times \square^n$$

*is equidimensional over  $\square^n$  of relative dimension  $t$ , and has modulus  $Y$ . In fact, the defining polynomials  $\Phi_i^m$  can be taken to have degree  $\leq 2$  for each variable  $y_j$ .*



It is proved by induction on  $m$ , starting with  $\Phi^0 = \text{id}$  which has degree 0.

### 3 Suslin homology with modulus and Higher Chow groups with modulus

In this section, let  $\overline{X}$  be an *affine* algebraic  $k$ -scheme and  $X$  be an open subset such that  $\overline{X} \setminus X$  is the support of an effective divisor. The aim of this section is to prove the following theorem:

**Theorem 3.1.** *The inclusions*

$$z_r^{\text{equi}}(\overline{X}|Y, \bullet) \subset z_r(\overline{X}|Y, \bullet)$$

*induce pro-isomorphisms on the homology groups:*

$$“\lim_Y” H_n(z_r^{\text{equi}}(\overline{X}|Y, \bullet)) \cong “\lim_Y” \text{CH}_r(\overline{X}|Y, n)$$

*where  $Y$  runs through effective Cartier divisors with support  $\overline{X} \setminus X$ .*

**Remark 3.2.** In the terminology of [FI, §6], the above theorem can be expressed as: the map “ $\lim_Y$ ”  $z_r^{\text{equi}}(\overline{X}|Y, \bullet) \rightarrow$  “ $\lim_Y$ ”  $z_r(\overline{X}|Y, \bullet)$  is a weak equivalence in the  $\mathcal{H}_*$ -model category of pro-complexes of abelian groups.

**Remark 3.3.** In fact, we prove below that the inclusions

$$\underline{z}_r^{\text{equi}}(\overline{X}|Y, \bullet) \subset \underline{z}_r(\overline{X}|Y, \bullet)$$

induce pro-isomorphisms on the homology groups

$$“\lim_Y” H_n(\underline{z}_r^{\text{equi}}(\overline{X}|Y, \bullet)) \cong “\lim_Y” H_n(\underline{z}_r(\overline{X}|Y, \bullet)).$$

Then, by the canonical splitting we saw in Remark 1.5, Theorem 3.1 is an immediate consequence of the last isomorphisms.

Theorem 3.1 is stated for a general base field. The proof can be easily reduced to the case over an infinite base field by a norm (trace) argument. In what follows, we will assume the base field  $k$  is infinite so that we may use the results of §2.

#### 3.1 Construction of weak homotopy

**Definition 3.4.** Let  $N$  be a positive integer. Suppose that for any  $0 \leq n \leq N$ , we are given a  $\overline{X}$ -morphism  $\varphi_n : \overline{X} \times \square^n \rightarrow \overline{X} \times \square^n$  such that for any  $0 \leq j \leq n \leq N$  the following diagram is commutative:

$$\begin{array}{ccc} \overline{X} \times \square^{n-1} & \xrightarrow{\varphi_{n-1}} & \overline{X} \times \square^{n-1} \\ \downarrow 1_{\overline{X}} \times s_j & & \downarrow 1_{\overline{X}} \times s_j \\ \overline{X} \times \square^n & \xrightarrow{\varphi_n} & \overline{X} \times \square^n. \end{array}$$

We define a subgroup  ${}_{\varphi}\underline{z}_r(\overline{X}|Y, n) \subset \underline{z}_r(\overline{X}|Y, n)$  to be the free abelian group on the set of integral closed subschemes  $V \subset X \times \square^n$  such that  $[V] \in \underline{z}_r(\overline{X}|Y, n)$  and the pullback  $\varphi_n^*[V]$  is defined and contained in  $\underline{z}_r(\overline{X}|Y, n)$ . Then,  ${}_{\varphi}\underline{z}_r(\overline{X}|Y, \bullet)$  defines a subcomplex of  $\underline{z}_r(\overline{X}|Y, \bullet)$ .

In the following, we fix a closed subscheme  $V \subset \overline{X} \times \square^n$  whose irreducible components have modulus  $2n \cdot Y$  and take  $\varphi_n := \Phi^n$ , where  $\Phi^\bullet$  is the system of morphisms given in Theorem 2.8.

**Definition 3.5.** Define for each  $n \geq 0$  an abelian subgroup  $\Phi \underline{z}_r^-(\overline{X}|Y, n) \subset {}_{\Phi}\underline{z}_r(\overline{X}|Y, n)$  by

$$\Phi \underline{z}_r^-(\overline{X}|Y, n) := {}_{\Phi}\underline{z}_r(\overline{X}|Y, n) \cap \underline{z}_r(\overline{X}|2nY, n) \subset {}_{\Phi}\underline{z}_r(\overline{X}|Y, n).$$

Then, we get a subcomplex  $\Phi \underline{z}_r^-(\overline{X}|Y, \bullet) \subset {}_{\Phi}\underline{z}_r(\overline{X}|Y, \bullet)$ .

**Lemma 3.6.** *The homomorphisms*

$$\Phi \underline{z}_r^-(\overline{X}|Y, \bullet) \xrightarrow[\text{incl.}]{(\Phi^\bullet)^*} \underline{z}_r(\overline{X}|Y, \bullet)$$

are weakly homotopic (i.e. their restriction to any finitely generated subcomplex are homotopic).

*Proof.* To construct a weak homotopy as in the assertion, we fix a finite set of integral closed subschemes  $\{V_k^n\} \subset X \times \square^n \in \Phi \underline{z}_r^-(\overline{X}|Y, n)$  which is closed under pullback along faces. Denote by  $C_n$  the free abelian group generated by  $[V_k^n]$ 's. Then, we get a subcomplex  $C_* \subset \Phi \underline{z}_r^-(\overline{X}|Y, \bullet)$ . Since the subcomplexes of this form are cofinal in all finitely generated subcomplexes, it suffices to prove that  $(\varphi - \text{incl.})|_{C_*}$  is homotopic to zero. For the proof, we construct a family of  $\overline{X}$ -morphisms  $\tilde{\Phi}^n : \overline{X} \times \square^n \times \mathbb{A}^1 \rightarrow \overline{X} \times \square^n \times \mathbb{A}^1$  which satisfies the following conditions:

(1) The following diagrams commute:

$$\begin{array}{ccc} \overline{X} \times \square^n & \xrightarrow{\text{id}} & \overline{X} \times \square^n \\ \downarrow i_0 & & \downarrow i_0 \\ \overline{X} \times \square^n \times \mathbb{A}^1 & \xrightarrow{\tilde{\Phi}^n} & \overline{X} \times \square^n \times \mathbb{A}^1, \\ \\ \overline{X} \times \square^n & \xrightarrow{\Phi^n} & \overline{X} \times \square^n \\ \downarrow i_1 & & \downarrow i_1 \\ \overline{X} \times \square^n \times \mathbb{A}^1 & \xrightarrow{\tilde{\Phi}^n} & \overline{X} \times \square^n \times \mathbb{A}^1, \\ \\ \overline{X} \times \square^{n-1} \times \mathbb{A}^1 & \xrightarrow{\tilde{\Phi}^{n-1}} & \overline{X} \times \square^{n-1} \times \mathbb{A}^1 \\ \downarrow 1_{\overline{X}} \times s_j \times 1_{\mathbb{A}^1} & & \downarrow 1_{\overline{X}} \times s_j \times 1_{\mathbb{A}^1} \\ \overline{X} \times \square^n \times \mathbb{A}^1 & \xrightarrow{\tilde{\Phi}^n} & \overline{X} \times \square^n \times \mathbb{A}^1. \end{array}$$

- (2) Set  $Z := (\square^n \times 0) + (\square^n \times 1) + \partial \square^n \times \mathbb{A}^1 \subset \square^n \times \mathbb{A}^1$ . Then, for any point  $z \in \square^n \times \mathbb{A}^1$  outside  $Z$ , the dimension of the fiber over  $z$  of the map  $(\tilde{\Phi}^n)^{-1}(\cup_k V_k^n \times \mathbb{A}^1) \subset \overline{X} \times \square^n \times \mathbb{A}^1 \rightarrow \square^n \times \mathbb{A}^1$  is  $\leq r$ .
- (3) Every component of  $(\tilde{\Phi}^n)^{-1}(V_k^n \times \mathbb{A}^1)$  has modulus  $Y$ .

Given  $\tilde{\Phi}^\bullet$  as above, we may define a homotopy  $\sigma$  as  $\sigma(V_k^n) := (\tilde{\Phi}^n)^*(V_k^n \times \mathbb{A}^1)$ .

Now we construct  $\tilde{\Phi}^\bullet$ . Actually each component of  $\tilde{\Phi}^n$  will have degrees  $\leq 2$  in each variable  $y_j$ , which implies the condition (3) by Proposition 2.4 applied to  $n' = n + 1$ . Suppose we have constructed  $\tilde{\Phi}^{n-1}$ . Via the isomorphism  $\square^n \times \mathbb{A}^1 \cong \square^{n+1}$ , we have  $Z \cong \partial \square^{n+1}$ . Condition (1) for  $\tilde{\Phi}^{n-1}$  implies that there is a glued  $\overline{X}$ -map

$$\overline{X} \times Z \rightarrow \overline{X} \times \square^n \times \mathbb{A}^1,$$

whose restrictions to codimension 1 faces of  $\square^n \times \mathbb{A}^1 \cong \square^{n+1}$  are either  $\text{id}$ ,  $\Phi^n$  or  $\tilde{\Phi}^{n-1}$ . By the induction hypothesis and Theorem 2.8, these are defined by polynomials whose degrees in  $y_j$  are  $\leq 2$  for each  $j$ . Then by Theorem 2.2, we obtain  $\tilde{\Phi}^n$  having degrees  $\leq 2$  and satisfying (1)(2).  $\square$

### 3.2 Proof of the comparison theorem

Finally we can prove Theorem 3.1. In the following, we use the following abbreviations:

$$C_\bullet^Y := \underline{z}_r^{\text{equi}}(\overline{X}|Y, \bullet), \quad D_\bullet^Y := \underline{z}_r(\overline{X}|Y, \bullet),$$

Let  $f^Y : C_\bullet^Y \rightarrow D_\bullet^Y$  denote the natural inclusion

By Remark 3.3, it suffices to prove that “ $\lim_Y$ ”  $H_n C_\bullet^Y \xrightarrow{\text{“lim” } H_n f^Y} \text{“lim}_Y H_n D_\bullet^Y$ ” is an isomorphism in the category of pro-abelian groups  $\text{pro-Ab}$ . Since the functor “ $\lim$ ” is exact, the kernel and the cokernel of the map “ $\lim$ ”  $H_n f^Y$  is given by “ $\lim$ ”  $\text{Ker}(H_n f^Y)$ , “ $\lim$ ”  $\text{Coker}(H_n f^Y)$ . We prove that these objects in  $\text{pro-Ab}$  are the zero object. Now we have the following elementary lemma:

**Lemma 3.7.** *An object  $A = \{A^\gamma\}_{\gamma \in \Gamma} \in \text{pro-Ab}$  is the zero object if and only if for any  $\gamma \in \Gamma$  there exists  $\gamma' > \gamma$  such that the projection map  $p_{\gamma'}^{\gamma'} : A^{\gamma'} \rightarrow A^\gamma$  is the zero map.*

Therefore, we are reduced to showing the following

**Lemma 3.8.** *For any principal effective divisor  $Y$  and  $n \geq 0$ , there exists  $N > 1$  such that the projections  $\text{Ker}(H_n f^{NY}) \rightarrow \text{Ker}(H_n f^Y)$  and  $\text{Coker}(H_n f^{NY}) \rightarrow \text{Coker}(H_n f^Y)$  are the zero maps.*

*Proof.* We firstly prove that  $\text{Coker}(H_n f^{2nY}) \rightarrow \text{Coker}(H_n f^Y)$  is the zero map for any  $n \geq 0$ . Take arbitrary element  $W \in H_n D_\bullet^{2nY}$ . Applying Lemma 3.6 for  $\Phi$  given in Theorem 2.8 with respect to  $W$ , there exists  $V \in C_n^Y$  such that  $\text{pr}_Y^{2nY} W = f^Y V$  holds in  $H_n D_\bullet^Y$ . This means that  $(H_n D_\bullet^{2nY} \rightarrow \text{Coker}(H_n f^{2nY}) \rightarrow \text{Coker}(H_n f^Y))$  is the zero map.

Next we prove that  $\text{Ker}(H_n f^{(2n+2)Y}) \rightarrow \text{Ker}(H_n f^Y)$  is the zero map. Take  $V \in \text{Ker}(H_n f^{(2n+2)Y}) \subset H_n C_{\bullet}^{(2n+2)Y}$  arbitrarily. In the following, we regard  $V$  as an element of  $C_n^{(2n+2)Y} \xrightarrow{f^{(2n+2)Y}} D_n^{(2n+2)Y}$ . Then, there exists  $W \in D_{n+1}^{(2n+2)Y}$  such that  $V = dW$  holds in  $D_n^{(2n+2)Y}$ . It suffices to show that  $\text{pr}_Y^{(2n+2)Y} dW$  belongs to  $dC_{n+1}^Y$ .

Let  $\Phi$  be the morphism as in Theorem 2.8 corresponding to  $W$ . Since we have  $(\Phi^{n+1})^* W \in C_{n+1}^Y$  by Theorem 2.8, it is equivalent to verify  $d(\Phi^{n+1})^* W - dW \in dC_{n+1}^Y$ . This element can be rewritten as

$$\begin{aligned} & d((\Phi^{n+1})^* - \text{incl.})W \\ &= d(d\sigma_{n+1} - \sigma_n d)W \\ &= -d\sigma_n dW \\ &= -d(\tilde{\Phi}^n)^*(d(W) \times \mathbb{A}^1), \end{aligned}$$

where  $\tilde{\Phi}$  and  $\sigma$  are defined in the proof of Lemma 3.6. By construction of  $\tilde{\Phi}$ , we can see that  $(\tilde{\Phi}^n)^*(d(W) \times \mathbb{A}^1)$  is equidimensional. Therefore, the right hand side of the equations belongs to  $dC_{n+1}^Y$ , which proves the desired assertion.  $\square$

### 3.3 A consequence on the relative motivic cohomologies

We can naturally sheafify our objects and consider the inclusion

$$z_r^{\text{equi}}(\overline{X}|Y, \bullet)_{\text{Zar}} \subset z_r(\overline{X}|Y, \bullet)_{\text{Zar}}$$

of Zariski sheaves of complexes on  $\overline{X}$ . The induced maps on homology sheaves

$$“\lim_Y” H_n(z_r^{\text{equi}}(\overline{X}|Y, \bullet)_{\text{Zar}}) \xrightarrow{“\lim_Y” f_n^Y} “\lim_Y” \text{CH}_r(\overline{X}|Y, n)_{\text{Zar}}$$

are pro-isomorphisms of Zariski sheaves for all  $n$ . Indeed, by Lemma 3.8, the maps of sheaves

$$\begin{aligned} \text{Coker}(f_n^{2nY}) &\rightarrow \text{Coker}(f_n^Y) \\ \text{Ker}(f_n^{(2n+2)Y}) &\rightarrow \text{Ker}(f_n^Y) \end{aligned}$$

are zero.

As a general fact on pro-categories, the functors  $H_{\text{Zar}}^n(\overline{X}, -)$  extend to functors

$$\begin{aligned} \text{pro-sheaves} &\rightarrow \text{pro-abelian groups} \\ “\lim_i” F_i &\mapsto “\lim_i” H_{\text{Zar}}^n(\overline{X}, F_i). \end{aligned}$$

We have hypercohomology spectral sequences in the abelian category of pro-abelian groups:

$$\begin{aligned} E_2^{pq} &= “\lim_Y” H_{\text{Zar}}^p(\overline{X}, H_{-q}(z_r^{\text{equi}}(\overline{X}|Y, \bullet)_{\text{Zar}})) \Rightarrow “\lim_Y” \mathbf{H}_{\text{Zar}}^{p+q}(\overline{X}, z_r^{\text{equi}}(\overline{X}|Y, \bullet)_{\text{Zar}}) \\ {}'E_2^{pq} &= “\lim_Y” H_{\text{Zar}}^p(\overline{X}, CH_r(\overline{X}|Y, -q)_{\text{Zar}}) \Rightarrow “\lim_Y” \mathbf{H}_{\text{Zar}}^{p+q}(\overline{X}, z_r(\overline{X}|Y, \bullet)_{\text{Zar}}) \end{aligned}$$

which are bounded to the range  $0 \leq p \leq \dim \overline{X}$  and  $q \leq 0$ . Since the natural map  $E \rightarrow {}'E$  of spectral sequences induces isomorphisms on  $E_2$ -terms, we get isomorphisms

$${}^{\text{“}}\lim_Y {}^{\text{”}} \mathbf{H}_{\text{Zar}}^n(\overline{X}, z_r^{\text{equi}}(\overline{X}|Y, \bullet)_{\text{Zar}}) \rightarrow {}^{\text{“}}\lim_Y {}^{\text{”}} \mathbf{H}_{\text{Zar}}^n(\overline{X}, z_r(\overline{X}|Y, \bullet)_{\text{Zar}}).$$

So we have proved:

**Theorem 3.9.** *For any algebraic scheme  $\overline{X}$  and an effective Cartier divisor  $Y_0$  on  $\overline{X}$ , the natural maps of pro-abelian groups*

$${}^{\text{“}}\lim_Y {}^{\text{”}} \mathbf{H}_{\text{Zar}}^n(\overline{X}, z_r^{\text{equi}}(\overline{X}|Y, \bullet)_{\text{Zar}}) \rightarrow {}^{\text{“}}\lim_Y {}^{\text{”}} \mathbf{H}_{\text{Zar}}^n(\overline{X}, z_r(\overline{X}|Y, \bullet)_{\text{Zar}})$$

*are isomorphisms, where  $Y$  runs through effective Cartier divisors with support  $|Y_0|$ .*

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